Ex:- Examine the continuity of the function defined by

$$f(x) = \frac{|x-a|}{x-a}, \quad x \neq a$$
$$= 1, \quad x = a$$

at the point $x = a$

Sol^n

$$\lim_{x \to a+0} f(x) = \lim_{x \to a+0} \left(\frac{x-a}{x-a}\right) = 1,$$

$$\lim_{x \to a-0} f(x) = \lim_{x \to a-0} \left(-\frac{(x-a)}{x-a}\right) = -1$$

$|x-a| = x-a, \; x > a$

$|x-a| = -(x-a), \; x < a$

$$f(a) = 1$$

$$\Rightarrow \lim_{x \to a+0} f(x) = f(a) \neq \lim_{x \to a-0} f(x).$$

Therefore f has a discontinuity of the first kind from the left at $x = a$.

Ex:- Discuss the continuity of the function $f(x) = [x] + [-x]$ at integral values of x.

Sol^n i) If x is an integer,

$[x] = x$ and $[-x] = -x \Rightarrow f(x) = 0$

ii). If x is not an integer,

Let $x = n + f$ where n is an integer and $f \in (0,1)$.

$\Rightarrow [x] = n$ and $[-x] = [-n-f]$

$= [(-n-1) + (1-f)] = (-n-1)$

$\therefore f(x) = n + (-n-1) = -1$

$$\therefore f(x) = \begin{cases} 0, & \text{if } x \text{ is an integer} \\ -1, & \text{if } x \text{ is not an integer} \end{cases}$$

At $x=a$, where a is an integer

$$\lim_{x \to a-0} f(x) = -1$$

and $\lim_{x \to a+0} f(x) = 1$ (As $a+0$ and $a-0$ are not integers)

but $f(a) = 0$ and a is an integer

Hence, $f(x)$ has a removable discontinuity at integral values of x.

Ex 1) Prove that the function f defined as

$$f(x) = \begin{cases} x, & x \le 1 \\ 2-x, & 1 < x \le 2 \\ -2+3x-x^2, & x > 2 \end{cases}$$

is continuous at $x=1$ and $x=2$.

2) A function $f(x)$ is defined by

$$f(x) = \begin{cases} \dfrac{[x^2]-1}{x^2-1}, & \text{for } x^2 \neq 1 \\ 0, & \text{for } x^2 = 1 \end{cases}$$

Discuss the continuity of $f(x)$ is continuous at $x=1$.

3) If $f(x) = \dfrac{\sin 2x + A\sin x + B\cos x}{x^3}$ is continuous at $x=0$, find the values of A, B and $f(0)$.

Th Limit of a function, if exists, is unique.

Proof If possible, let l and l' be the two distinct limits of f at c. Then for every arbitrarily chosen ε, $0<\varepsilon<\frac{1}{2}|l-l'|$, we have a $\delta_\varepsilon>0$ such that

$$|f(x)-l|<\varepsilon \text{ whenever } 0<|x-c|<\delta_\varepsilon \quad \text{— (i)}$$

and $|f(x)-l'|<\varepsilon$ whenever $0<|x-c|<\delta_\varepsilon$ — (ii)

Therefore, from (i) and (ii) we have for $0<|x-c|<\delta_\varepsilon$,

$$|l-l'| = |l-f(x)+f(x)-l'| \leq |f(x)-l|+|f(x)-l'|$$
$$<\varepsilon+\varepsilon = 2\varepsilon < |l-l'|$$

— which is a contradiction.

Hence, the theorem is proved.

Th If $f(x)$ tends to a finite limit as $x \to a$ then there is a deleted neighbourhood of a in which f is bounded.

Proof Let $\lim_{x\to a} f(x) = l$. Then $\exists$ a $\delta>0$ such that

$$|f(x)-l|<1 \quad \forall x: 0<|x-a|<\delta$$

$$\Rightarrow l-1<f(x)<l+1 \quad \forall x \in (a-\delta, a+\delta)\setminus\{a\}.$$

Hence, the theorem follows.

Th If $\lim_{x\to a} f(x) = l$ then $\lim_{x\to a} |f(x)| = |l|$

Proof If $\lim_{x\to a} f(x) = l$, then for every $\varepsilon>0$ there corresponds a $\delta>0$ such that

$$|f(x)-l|<\varepsilon \quad \forall x: 0<|x-a|<\delta \quad \text{— (i)}$$

Since $||f(x)| - |l|| \le |f(x) - l|$ so it follows from (i)

$$||f(x)| - |l|| < \varepsilon \quad \forall x : 0 < |x-a| < \delta$$

Hence, $\lim_{x \to a} |f(x)| = |l|$.

<u>Remark</u> The converse of the above theorem is not always true.

Let $f(x) = 1$ for rational $x \in \mathbb{R}$
$\quad\quad\; = -1$ for irrational $x \in \mathbb{R}$

Then $\lim_{x \to 0} f(x)$ does not exist but $\lim_{x \to 0} |f(x)| = 1$.

<u>Defn</u> <u>Sequential defn of limit</u>:

Let $f : I \to \mathbb{R}$, where I is an interval, and let $a \in \bar{I}$, the closure of I. Then, if $\exists$ a real number l such that for every sequence $\{x_n\} \subset I$ with $\lim_{n \to \infty} x_n = a$ we have

$$\lim_{n \to \infty} f(x_n) = l.$$

<u>Th</u> A function $f : I \to \mathbb{R}$, where I is an interval, has a limit l as $x \to a \in \bar{I}$, iff for every sequence $\{x_n\} \subset I$, $\lim_{n \to \infty} x_n = a$, we have $\lim_{n \to \infty} f(x_n) = l$.

<u>Proof</u> Let $\lim_{x \to a} f(x) = l$, ~~let~~ & let $\varepsilon > 0$ be arbitrary.

Then $\exists\ \delta > 0$ such that

$$|f(x) - l| < \varepsilon \quad \forall x \in I : 0 < |x-a| < \delta \quad \text{——(i)}$$

Let $\{x_n\}$ be any sequence in I such that $\lim_{n \to \infty} x_n = a$. Then there is a positive integer

N such that for $n \geq N$, $|x_n - a| < \delta$, and hence from (i) it follows that $|f(x_n) - l| < \varepsilon$ $\forall n \geq N$. Hence

$$\lim_{n\to\infty} f(x_n) = l.$$

Conversely, let for every sequence $\{x_n\} \subset I$ and

$$\lim_{n\to\infty} x_n = a \Rightarrow \lim_{n\to\infty} f(x_n) = l.$$

Let us suppose that $\lim_{x\to a} f(x) \neq l$.

Then $\exists$ an ε_0 such that for every $\delta' > 0$ there is $x' \in I$ such that $|x' - a| < \delta'$ and

$$|f(x') - l| \geq \varepsilon_0 \quad \text{——— (ii)}$$

Let $\{\delta_n\}$ be a sequence of reals such that $0 < \delta_{n+1} < \delta_n$ $\forall n$ and $\lim_{n\to\infty} \delta_n = 0$.

Then $\exists$ a sequence $\{x_n\} \subset I$ such that $|x_n - a| < \delta_n$ and $|f(x_n) - l| \geq \varepsilon_0$ $\forall n$.

Hence, we have $\lim_{n\to\infty} x_n = a$ but $\lim_{n\to\infty} f(x_n) \neq l$.

— which is a contradiction.

Hence the theorem follows.

Ex:- S.T. $\lim_{x\to 0} \frac{1}{x}$ does not exist.

Sol^n We choose $\{x_n\} = \{\frac{1}{n}\}$ and $\{x_n'\} = \{-\frac{1}{n}\}$. Then

$$\lim_{x_n\to 0} \frac{1}{x_n} = \lim_{n\to\infty} n = \infty \quad \text{and}$$

$$\lim_{x_n'\to 0} \frac{1}{x_n'} = \lim_{n\to\infty} (-n) = -\infty.$$

Hence, $\lim_{x\to 0} \frac{1}{x}$ does not exist.

Cauchy's General Principle for the existance of limit:

Th Let $f: [a,b] \to \mathbb{R}$ and let $c \in [a,b]$. Then a necessary and sufficient condition that $f(x)$ tends to a limit as $x \to c$ is that for any $\varepsilon > 0$ there is $\delta > 0$ such that for $x', x'' \in (a,b)$,

$$|f(x') - f(x'')| < \varepsilon \text{ whenever } 0 < |x'-c| < \delta$$
$$\text{and } 0 < |x''-c| < \delta.$$

Proof Let $\lim_{x \to c} f(x) = l$. Let $\varepsilon > 0$ be arbitrary. Then $\exists$ a $\delta > 0$ such that

$$|f(x) - l| < \frac{\varepsilon}{2} \text{ whenever } 0<|x-c|<\delta \quad \text{---} \rightarrow (i)$$

Let x' and $x'' \in (a,b)$ and $0 < |x'-c| < \delta$ and $0<|x''-c|<\delta$.

Then from (i),

$$|f(x') - l| < \varepsilon/2 \text{ and } |f(x'') - l| < \varepsilon/2 \quad \text{---} \rightarrow (ii)$$

Hence,

$$|f(x') - f(x'')| = |f(x') - l + l - f(x'')| < |f(x')-l| + |f(x'')-l|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon \text{ [by (ii)]}$$

$\therefore$ $|f(x') - f(x'')| < \varepsilon$ whenever $|x'-c|<\delta$ and $|x''-c|<\delta$.

Thus the condition is necessary.

Conversely, let for $\varepsilon > 0$, $\exists$ $\delta > 0$ such that for each pair $x', x'' \in (a,b)$ and $c \in [a,b]$,

$$|f(x') - f(x'')| < \varepsilon \text{ whenever } 0<|x'-c|<\delta \text{ and } 0<|x''-c|<\delta \quad \text{---} \rightarrow (iii)$$

Let $\{x_n\}$ be any sequence in (a,b) and let $\lim_{n\to\infty} x_n = c$.

Then we have an integer N, such that

$0 < |x_n - c| < \delta$ for $n \geq N$ and so by (iii) we get

$$|f(x_n) - f(x_m)| < \varepsilon \text{ for } n, m \geq N.$$

So $\{f(x_n)\}$ is a Cauchy sequence and hence $\{f(x_n)\}$ is convergent.

Let $\lim_{n\to\infty} f(x_n) = l$.

Now, $\forall\, x \in (a,b)$ and $x_n \in \{x_n\}$ such that $0<|x-c|<\delta$ and $0<|x_n-c|<\delta$, we have by (iii),

$$|f(x) - f(x_n)| < \varepsilon.$$

So, $\lim_{n\to\infty} |f(x) - f(x_n)| \leq \varepsilon$

i.e. $|f(x) - \lim_{n\to\infty} f(x_n)| \leq \varepsilon$

i.e. $|f(x) - l| \leq \varepsilon$ when $0<|x-c|<\delta$.

Hence, $\lim_{x\to c} f(x) = l$. and hence the theorem follows.

Algebra of Limits

Th Let $\lim_{x\to a} u_1 = l_1$ and $\lim_{x\to a} u_2 = l_2$, then

$$\lim_{x\to a} (u_1(x) + u_2(x)) = \lim_{x\to a} u_1(x) + \lim_{x\to a} u_2(x) = l_1 + l_2.$$

Proof ~~Let~~ Here $\lim_{x\to a} u_1(x) = l_1$ and $\lim_{x\to a} u_2(x) = l_2$.

Let $\varepsilon>0$ be chosen arbitrarily.

Then for above chosen $\varepsilon>0$, $\exists\ \delta_1, \delta_2>0$ such that

$$|u_1 - l_1| < \varepsilon/2 \ \forall x \text{ such that } 0<|x-a|<\delta_1 \quad \text{--- (1)}$$

and $|u_2 - l_2| < \varepsilon/2 \ \forall x$ such that $0<|x-a|<\delta_2$ --- (2)

Now, $|(u_1+u_2) - (l_1+l_2)| \leq |u_1 - l_1| + |u_2 - l_2|$ --- (3)

Let $\delta = \min[\delta_1, \delta_2]$. Then from (1), (2) and (3) we get,

$|(u_1+u_2) - (l_1+l_2)| < \varepsilon/2 + \varepsilon/2 = \varepsilon\ \forall x$ s.t. $0<|x-a|<\delta$.

Hence, $\lim_{x\to a} [u_1(x) + u_2(x)] = \lim_{x\to a} u_1(x) + \lim_{x\to a} u_2(x) = l_1 + l_2$

[Proved]

Th If $\lim_{x\to a} u_1(x) = l_1$ and $\lim_{x\to a} u_2(x) = l_2$,

then $\lim_{x\to a} [u_1(x) - u_2(x)] = l_1 - l_2$.

Proof Do it by yourself.

Th If $\lim_{x\to a} u_1(x) = l_1$, $\lim_{x\to a} u_2(x) = l_2$, then

$$\lim_{x\to a} u_1(x)\, u_2(x) = l_1 l_2.$$

Proof Here $\lim_{x\to a} u_1(x) = l$, $\lim_{x\to a} u_2(x) = l_2$.

Let $\varepsilon > 0$ be chosen arbitrarily.

Then for every $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$|u_1(x) - l_1| < \varepsilon/2 \quad \forall x : 0 < |x-a| < \delta \quad \text{—— (1)}$$

and $$|u_2(x) - l_2| < \varepsilon/2 \quad \forall x : 0 < |x-a| < \delta \quad \text{—— (2)}$$

Then $$|u_1(x) . u_2(x) - l_1 l_2| = |u_1(x)\, u_2(x) - l_1 u_2(x) + l_1 u_2(x) - l_1 l_2|$$

$$\leq |u_2(x)|\,|u_1(x) - l_1| + |l_1|\,|u_2(x) - l_2| \quad \text{—— (3)}$$

and $$|u_2(x)| \leq |u_2(x) - l_2| + |l_2| \leq \varepsilon/2 + |l_2| \quad \forall x : 0 < |x-a| < \delta. \quad \text{(4)}$$

$\therefore$ From (1), (2), (3) and (4) it follows that

$$|u_1(x)\, u_2(x) - l_1 l_2| \leq (\varepsilon/2 + |l_2|)\varepsilon/2 + |l_1|\,\varepsilon/2$$

$$= (\varepsilon/2 + |l_2| + |l_1|)\,\varepsilon/2$$

$$< (1 + |l_1| + |l_2|)\,\varepsilon \quad \forall x : 0 < |x-a| < \delta$$

$\because$ $\varepsilon > 0$ is arbitrary, so we have

$$\lim_{x\to a} u_1(x)\, u_2(x) = l_1 l_2.$$

Hence, the theorem is proved.

Th Let $u_1(x)$ and $u_2(x)$ be two functions such that $\lim_{x\to a} u_1(x) = l$ and $\lim_{x\to a} u_2(x) = m$ and $m \neq 0$.

Then $\lim_{x\to a} \dfrac{u_1(x)}{u_2(x)} = \dfrac{l}{m}$ (provided $m \neq 0$).

Proof Let $\varepsilon > 0$ be chosen arbitrarily.

Then $\exists\ \delta_1, \delta_2, \delta_3 > 0$ such that

$|u_1(x) - l| < \varepsilon$ for $0 < |x-a| < \delta_1$ —— (1)

$|u_2(x) - m| < \varepsilon$ for $0 < |x-a| < \delta_2$ —— (2)

$|u_2(x) - m| < \frac{1}{2}|m|$ for $0 < |x-a| < \delta_3$ —— (3)

From (3)

$$|u_2(x)| = |m + u_2(x) - m| \geq |m| - |u_2(x) - m|$$
$$\geq |m| - \tfrac{1}{2}|m| = \tfrac{1}{2}|m| \text{ for } 0 < |x-a| < \delta_3 \quad \text{(4)}$$

So, if $\delta = \min[\delta_1, \delta_2, \delta_3]$ then from (1), (2) and (4)

$$\left|\frac{u_1(x)}{u_2(x)} - \frac{l}{m}\right| = \left|\frac{m\,u_1(x) - l\,u_2(x)}{m\,u_2(x)}\right|$$
$$= \left|\frac{m(u_1(x) - l) - l(u_2(x) - m)}{m\,u_2(x)}\right|$$
$$\leq \left|\frac{u_1(x) - l}{u_2(x)}\right| + \frac{|l|}{|m|}\left|\frac{u_2(x) - m}{u_2(x)}\right|$$
$$< \frac{2}{|m|}\varepsilon + \frac{|l|}{|m|}\cdot\frac{2}{|m|}\varepsilon = \frac{2}{|m|}\left(1 + \frac{|l|}{|m|}\right)\varepsilon$$

for $0 < |x-a| < \delta$

$\because\ \varepsilon > 0$ is arbitrary, so the theorem is proved.

Sandwitch Theorem

If $f(x) \le g(x) \le h(x)$ $\forall x$ in a neighbourhood of a and if $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = l$, then $\lim_{x\to a} g(x) = l$.

Proof Let $\delta_1 > 0$ be such that

$$f(x) \le g(x) \le h(x) \;\forall\; x : |x-a| < \delta_1 \quad \text{——} (i)$$

Case-I Let l be finite and let $\varepsilon > 0$ be arbitrary.

Then $\exists$ $\delta_2, \delta_3 > 0$ such that

$$l-\varepsilon < f(x) < l+\varepsilon \text{ for } 0<|x-a|<\delta_2 \quad \text{——} (ii)$$

$$l-\varepsilon < h(x) < l+\varepsilon \text{ for } 0<|x-a|<\delta_3 \quad \text{——} (iii)$$

Let $\delta = \min[\delta_1, \delta_2, \delta_3]$. Then from (i), (ii), (iii)

$$l-\varepsilon < f(x) \le g(x) \le h(x) < l+\varepsilon \text{ for } 0<|x-a|<\delta$$

and so

$$|g(x) - l| < \varepsilon \text{ for } 0<|x-a|<\delta.$$

Hence, $\lim_{x\to a} g(x) = l$.

Case-II If $l = \infty$, then for large positive number G, $\exists$ $\delta_2 > 0$, such that

$$f(x) > G \;;\; \forall x : 0<|x-a|<\delta_2 \quad \text{——} (iv)$$

Then from (i) and (iv) we have,

$$g(x) > G, \;\forall\; x : 0<|x-a|<\delta = \min\{\delta_1, \delta_2\}$$

which implies $\lim_{x\to a} g(x) = \infty$.

Case-III If $l = -\infty$, then for every large no. $G > 0$, $\exists$ $\delta_2 > 0$ such that

$$h(x) < -G \quad \forall\; x : 0<|x-a|<\delta_2 \quad \text{——} (v)$$

Hence, from (i) and (v) it follows that

$$g(x) < -G, \;\forall x : 0<|x-a|<\delta = \min\{\delta_1, \delta_2\}$$

$$\Rightarrow \lim_{x\to a} g(x) = -\infty.$$

Hence the theorem follows.